Symmetric closure in modules and rings

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2 Symmetric groups and factorizations

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3 Symmetrically closed graphs and their diameters

Definition

- Let M be a right R-module. Two elements $m, n \in M$ are *symmetrically connected* if there exist $m' \in M$ and $a, b \in R$ such that $m = m'ab$ and $n = m'ba$. We denote this situation by $m \overset{1}{\sim} n$.
- Two elements $m, n \in M$ are *symmetrically related* if there exists a finite chain of symmetrically connected elements $m = m_0 \overset{1}{\sim} m_1 \overset{1}{\sim} \cdots \overset{1}{\sim} m_\ell = n$. We will write $m \sim n$ when m and n are symmetrically related.
- For an element $m \in M$, we put $\widehat{\{m\}} := \{n \in M \mid n \sim m\}$, and we say that an element $m \in M$ is *symmetrically closed* when $\widehat{\{m\}} = \{m\}$.

Example

1. If R is a commutative ring, then two elements of a module M_R are symmetrically connected if and only if they are equal.
2. Recall that a ring R is said to be *symmetric* if and only if for any $a, b, c \in R$, $abc = 0$ implies $acb = 0$. If $M = R_R$, then $\widehat{\{0\}} = \{0\}$ if and only if the ring R is symmetric.
3. For a ring R, Let $U(R)$ and $U(R)' = \langle [u, v] = uvu^{-1}v^{-1} : u, v \in R \rangle$ denote the group of *units* of R and its *derived group*, respectively. If $m \in M_R$, then we always have $m(U(R))' \subseteq \widehat{\{m\}}$; indeed, if $u, v \in U(R)$, then $m[u, v] = muvu^{-1}v^{-1} \overset{1}{\sim} muvv^{-1}u^{-1} = m$.

Definition

- Let m be a nonzero element in a module M_R. Also, let $\mathrm{rann}(r) := \{s \in R \mid rs = 0\}$ and $\mathrm{ann}(m) = \{x \in R \mid mx = 0\}$. We say that *r divides m* if $\mathrm{rann}(r) \subseteq \mathrm{ann}(m)$ and there exists $m' \in M$ such that $m = m'r$.
- Recall that an element r in a ring R is called *regular* if there exists x in R such that $r = rxr$. We denote the set of regular elements by $Reg(R)$.

Proposition

Let $m \in M_R$ be a nonzero element of a right R-module. Suppose $r \in Reg(R)$ is such that $\mathrm{rann}(r) \subseteq \mathrm{ann}(m)$. Then $mx\widehat{\{r\}} \subseteq \widehat{\{m\}}$, where x is any quasi inverse of r, that is, we have $r = rxr$.

Definition

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Proposition

If $p \in M$ is an atom, then we have $p(U(R))' = \widehat{\{p\}}$.

Definition

Let R be a unitary ring and $a, b \in R$. We write:

- - $a \overset{c}{\sim}_1 b$ if there exist $x, y \in R$ such that $a = xy$ and $b = yx$.
  - $a \overset{*}{\sim}_1 b$ if there exist $x, y, z \in R$ such that $a = xyz$ and $b = xzy$.
  - $a \widehat{\sim}_1 b$ if there exist $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_n \in R$ and $\pi \in S_n$ such that $a = x_1 x_2 \cdots x_n$ and $b = x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)}$.
- For $s \in \{c, *, \wedge\}$, we define
  - $a \overset{s}{\sim}_\ell b$ if there exist $x_1, x_2, \ldots, x_\ell \in R$ such that
$$a \overset{s}{\sim}_1 x_1 \overset{s}{\sim}_1 x_2 \overset{s}{\sim}_1 \cdots \overset{s}{\sim}_1 x_\ell = b.$$
  - $a \overset{s}{\sim} b$ if there exists $\ell \in \mathbb{N}$ such that $a \overset{s}{\sim}_\ell b$.
  - $\{a\}^s = \{b \in R \mid a \overset{s}{\sim} b\}$. We also write $\overline{\{a\}}$ for $\{a\}^c$ and $\widehat{\{a\}}$ for $\{a\}^\wedge$.

Definition

- Let S be a subset of a ring R. We define the *symmetric closure* of S as $\widehat{S} = \bigcup_{s \in S} \widehat{\{s\}}$.
- S is called *symmetrically closed* if $S = \widehat{S}$.
- For $s \in \{c, *, \wedge\}$, we define

$$S_n^s = \{x \in R \mid \exists\, x_0 \in S \text{ such that } x \overset{s}{\sim}_n x_0\ \}.$$

In particular, for any $s \in S$, we have $\{s\}^* = \bigcup_{n \geq 0} \{s\}_n^*$.

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Proposition

Let $S \subseteq R$ be a subset of a Dedekind-finite ring R. Then the following statements hold:

- If S is a group, then $\widehat{S}$ is a group as well.
- $\{1\}_n^*$ is the set of products of at most n commutators.
- The closed set $\widehat{\{1\}}$ is the derived group $U(R)'$ of the group of units of R.
- If $S \subseteq U(R)$, then $\widehat{S} = S(U(R))'$.

Corollary

Let R be a ring. Then the following statements are equivalent:

- R is Dedekind-finite.
- $\overline{\{1\}} = \{1\}$.
- $\widehat{\{1\}} = U(R)'$.

Moreover, when R is Dedekind-finite, we have for any $a \in U(R)$, $\widehat{\{a\}} = a\widehat{\{1\}}$.

Example

Let $\mathbb{H}$ denote the division ring of *real quaternions*. For $x = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$ we define $N(x) = a_0^2 + a_1^2 + a_2^2 + a_3^2$. Moreover, let $\Gamma := \{x \in \mathbb{H} : N(x) = 1\}$. Then $\widehat{\{1\}} = \Gamma$.

Definition

- A ring R is called *reversible* if for any $a, b \in R$, we have $ab = 0$ implies that $ba = 0$.
- A ring R is said to be *semi-commutative* if for any $a, b \in R$, we have $ab = 0$ implies $aRb = 0$. Furthermore, any reversible ring is semi-commutative.

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Proposition

If R is semi-commutative, then $N(R)$ is symmetrically closed, where $N(R)$ denotes the set of nilpotent elements of the ring R. In particular, this holds if R is reversible.

Theorem

Let D be a division ring, $n \in \mathbb{N}$, $A \in M_n(D)$, and $GL_n(D)$ denotes the general linear group of non-singular $n \times n$ matrices with entries in D. Also, let I_n denote the identity matrix. Then the following statements hold:

- $\widehat{\{I_n\}} = GL_n(D)'$.
- If $A \in GL_n(D)$, then $\widehat{A} = A\widehat{\{I_n\}}$.
- If A is singular, then $\widehat{A} = \widehat{\{0\}}$.

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Lemma

Assume that R and S are two rings, and $(r, s) \in R \times S$. Then $\widehat{\{(r, s)\}} = \widehat{\{r\}} \times \widehat{\{s\}}$.

Definition

Let R be a unitary ring R and $s \in \{c, *, \wedge\}$.

- The elements of a class determined by $\overset{s}{\sim}$ can be seen as the set of vertices of a graph. Two elements x, y in the same class are said to be *adjacent* if $x \overset{s}{\sim}_1 y$.
- Let $x, y \in R$ be such that $x \overset{s}{\sim} y$, we define $d_s(x, y) = \min\{n \in \mathbb{N} \mid x \overset{s}{\sim}_n y\}$. We adopt the convention that $d_s(x, x) = 0$. It is not hard to check that d_s is a distance. This distance corresponds to the minimal length of the paths between two elements (vertices) in a class determined by $\overset{s}{\sim}$.
- For a subset S of R, we define

$$\mathrm{diam}_s(S) = \sup\{d_s(x, y) \mid x, y \in S \text{ and } x \overset{s}{\sim} y\}.$$

Theorem

Let R be a unitary ring. Then the following statements hold:

- If $t \in \widehat{\{z\}}$, then for any $m \in \mathbb{N}$, $t^m \in \widehat{\{z^m\}}$.
- A subset S of R is symmetrically closed and connected if and only if $S = \widehat{\{z\}}$ for some $z \in R$.
- For any subset S of R, $\mathrm{diam}_{\wedge}(S) \leq \mathrm{diam}_{\wedge}(\widehat{S})$ (respectively, $\mathrm{diam}_{*}(S) \leq \mathrm{diam}_{*}(\widehat{S})$).

Proposition

Let S be a subset of a ring R. Then the following statements hold:

- $\mathrm{diam}_*(S) \leq \mathrm{diam}_c(S)$. In particular, if $\mathrm{diam}_c(S)$ (respectively, $\mathrm{diam}_*(S)$) is finite (respectively, infinite), then $\mathrm{diam}_*(S)$ (respectively, $\mathrm{diam}_c(S)$) is finite (respectively, infinite).
- If R is a non-commutative Dedekind-finite, then $\mathrm{diam}_*(U(R)) = 1$. In particular, if D is a division ring, then $\mathrm{diam}_*(D) = 1$.

Proposition

Assume that z is an element in a ring R. If $n \in \mathbb{N}$ is the minimal number such that $\widehat{\{z\}} = \widehat{\{z\}}_n$, then $n \leq \mathrm{diam}_\wedge(\widehat{\{z\}}) \leq 2n$.

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Proposition

Let R and S be two rings. Also, let $\mathrm{diam}_\wedge(R) = n$ and $\mathrm{diam}_\wedge(S) = m$.
Then $\mathrm{diam}_\wedge(R \times S) = \max\{n, m\}$.
In addition, a similar result holds replacing $\mathrm{diam}_\wedge$ by diam_*.

Theorem

Let D be a division ring and $n \in \mathbb{N}$. Then $\mathrm{diam}_{\wedge}(M_n(D)) \leq 2$.

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Let F be a field and $n \in \mathbb{N}$. Then $\mathrm{diam}_{\wedge}(M_n(F)) = 1$.

Proposition

Let $n \in \mathbb{N}$, and let D be a division ring such that $n \neq 2$ and $D \neq \mathbb{F}_2$. Let $A, B \in GL_n(D)$ be two matrices such that $B \in \widehat{\{A\}}$. Let $SL_n(D)$ denote the special linear group of degree n over D, which is the set of $n \times n$ matrices with determinant 1. Then $AB^{-1} \in SL_n(D)$ and $d_*(A, B)$ is the minimal number of commutators required to express AB^{-1} as products of commutators in $GL_n(D)$.

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Proposition

- Let $F^2 = F \in M_n(D)$ be of rank k. Then $d_*(F, 0) \leq \lceil n/(n-k) \rceil$.
- Let $A, B \in M_n(D)$ be two singular matrices, and let A (respectively, B) be a product of $k \geq 1$ (respectively, $\ell \geq 1$) matrices similar to $E = \mathrm{diag}(1, \ldots, 1, 0)$. Then $d_*(A, B) \leq k + \ell$.
- $\mathrm{diam}_*(\widehat{\{0\}}) \leq 2n$.

Definition

- Recall that a *strictly upper triangular* matrix is an upper triangular matrix having 1's along the diagonal and 0's under it, i.e., a matrix $A = [a_{i,j}]$ such that $a_{i,j} = 0$ for all $i \geq j$ and $a_{ii} = 1$. We denote the set of all $n \times n$ strictly upper triangular matrix over a ring R by $U_n(R)$.
- For a ring R and $n \in \mathbb{N}$, we denote $N_n(R)$ as the set of elements of R that are nilpotent of index n.

Theorem

Let R be a ring. Then the following statements hold:

- If R is semi-commutative, then, for each $i \in \mathbb{N}$, we have $\{0\}_i^* \subseteq N_{2^i}(R)$. In particular, $\{0\}^* \subseteq N(R)$.
- For any strictly upper triangular matrix $U \in M_n(R)$, we have
  - (a) $U \in \widehat{\{0\}}_{n-1} \subseteq \widehat{\{0\}}$ and $U \in \{0\}_{n-1}^* \subseteq \{0\}^*$.
  - (b) $\mathrm{diam}_\wedge(U_n(R)) \leq 2(n-1)$ and $\mathrm{diam}_*(U_n(R)) \leq 2(n-1)$ for all $n \geq 2$.

Thank you for your attention